ON COMPOUND AND REVERSE CURVES.

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Curves consisting of a series of circular arcs each tangent to its predecessor are of constant use in the constructive problems of engineering and architecture; as for example for the centre lines of railways, highways, etc.; in the so-called basket-handled arches and in many of the arched forms of the pointed styles of architecture; in machine construction, and so on.

In the following note on these curves primary reference will be made to the location of curves for railways. But the results obtained will be applicable *mutatis mutandis* to every class of these problems.

We consider first the simple curve of a single circular arc. The curvature of these curves is measured most conveniently by the angle at the centre subtended by a chord of unit length. If we put (Fig. 1)

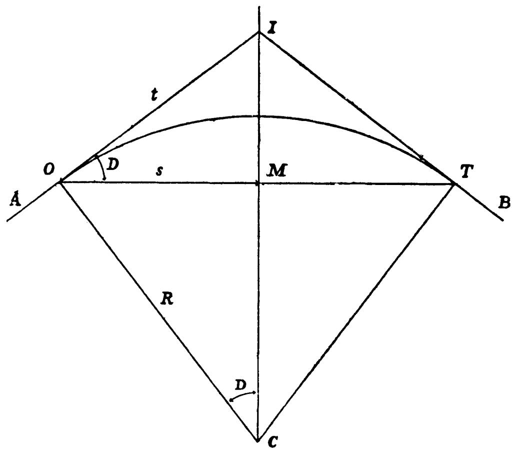

Fig. 1.

$R =$ the radius of the curve,
$\Delta =$ curvature of the curve as above defined,
$s =$ semi-chord of the arc,
$t =$ tangent measured from the point of contact to the intersection,
$D =$ angle between the chord and the tangent, called the deflection,

we shall have
$$\sin D = \frac{s}{R}, \quad \tan D = \frac{t}{R};$$

or, as $s = \frac{1}{2}$ when $D = \frac{1}{2}\Delta$,
$$\sin \tfrac{1}{2}\Delta = \frac{1}{2R};$$

whence, by eliminating R, we get
$$\sin D = 2s \sin \tfrac{1}{2}\Delta, \quad \tan D = 2t \sin \tfrac{1}{2}\Delta.$$

These are the fundamental relations by the aid of which all problems on simple curves are solved.

If the curvature Δ be given, then two other independent data fix the curve; as AI and BI; AI and O; O and T; and so on.

If the origin O be given, then as before, two other independent data fix the curvature and determine the curve. In this case, however, considerations of con-

venience confine us to integral values of Δ in minutes; and if necessary, the origin is shifted to conform to such a value. For example. if we have the length of the tangent OI given as 11.23, and the difference of direction of the tangents as $31^\circ\ 12'$, we have

$$D = 15^\circ\ 36', \quad 2t = 22.46;$$

$$\therefore \quad \sin \tfrac{1}{2}\Delta = \frac{\tan 15^\circ\ 36'}{22.46}$$

$$= [8.09451];$$

$$\therefore \quad \tfrac{1}{2}\Delta = 42'\ 44''.2, \text{ nearly.}$$

As it would be excessively inconvenient to have to have to set out such an angle repeatedly in the field, we take

$$\tfrac{1}{2}\Delta = 43';$$

$$\therefore \quad 2t = \frac{\tan 15^\circ\ 36'}{\sin 43'}$$

$$= [1.34874];$$

$$\therefore \quad t = 11.16,$$

and we have to shift O towards I through 0.07.

In the case of the simple curve the angles at O and T are necessarily equal. If these be unequal, we must have recourse to a compound curve of at least two circular arcs. Since the two arcs obey but five conditions,—pass through two points, touch two lines, and touch each other,—it is obvious that there is a sixth remaining condition, which can be arbitrarily imposed upon them.

If A_1J, A_2J (Fig. 2) be the two branches of such a curve, tangent at A_1, A_2 to A_1I, A_2I, and at J to the common tangent T_1T_2, it is easy to see that with the same notations as have been used for simple curves, $T_1 = 2D_1$, $T_2 = 2D_2$, and we have for the difference of azimuths I of the extreme tangents

$$A_1 + A_2 = 2D_1 + 2D_2,$$

which is the fundamental relation connecting the elements of the two arcs of a compound curve.

Since the external angle J of the triangle A_1JA_2 is equal to

$$D_1 + D_2, \quad \text{or} \quad \tfrac{1}{2}(A_1 + A_2),$$

and is therefore constant, the locus of the point J is a circular arc on A_1A_2 with deflection J and radius

$$r = \frac{c}{2 \sin J}$$

where $c = A_1A_2$

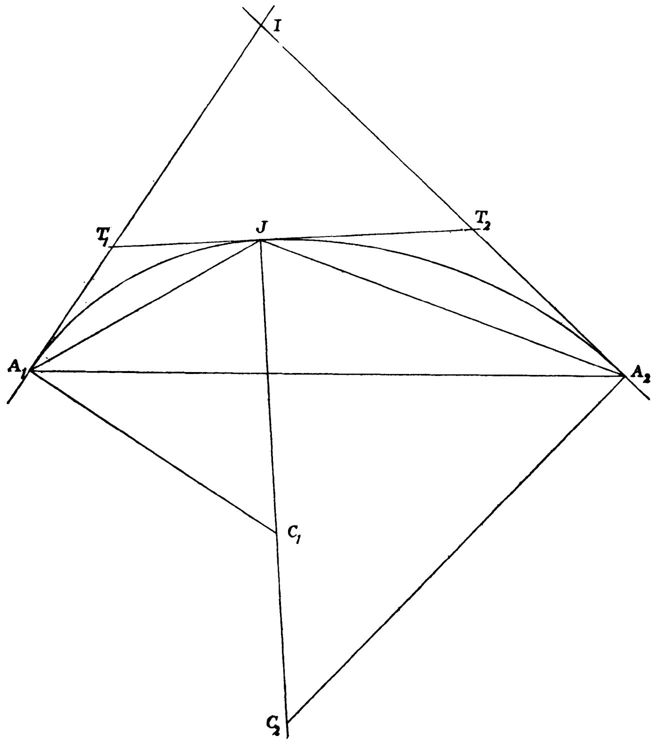

Fig. 2.

Again, since $$T_1J = T_1A_1, \quad T_2J = T_2A_2;$$
the perimeter of the triangle T_1IT_2 is

$$T_1I + T_2I + T_1T_2 = IA_1 + IA_2,$$

and is therefore constant. Accordingly, the envelope of T_1T_2 is a circular arc tangent to IA_1, IA_2 at points distant $\frac{1}{2}(IA_1 + IA_2)$ from I, whose radius, therefore, is

$$\rho = \frac{l_1 + l_2}{2 \tan J},$$

where $l_1 = IA_1$ and $l_2 = IA_2$.

The point J or T_1T_2 having been selected in accordance with either of these conditions, the two arcs of the compound curve are completely determined.

In fact, if we assign to D_1 (say) any arbitrary value, we have at once $D_2 = J - D_1$, and thence the angles $A_1 - D_1$, $A_2 - D_2$ of the triangle A_1JA_2; whence

$A_1J = 2s_1$ and $A_2J = 2s_2$ are determined, and from these the curvatures of the arcs are computed. In symbols

$$2s_1 = 2r\sin(A_2 - D_2), \quad 2s_2 = 2r\sin(A_1 - D_1);$$

$$\therefore \quad \sin\tfrac{1}{2}\Delta_1 = \frac{\sin D_1}{2r\sin(A_2 - D_2)}, \quad \sin\tfrac{1}{2}\Delta = \frac{\sin D_2}{2r\sin(A_1 - D_1)}.$$

Again, if M_1, M_2 be the points of contact of the arc whose radius is ρ,

$$T_1M_1 = \rho\tan D_1, \quad T_1A_1 = R_1\tan D_1;$$

whence, since $A_1M_1 = \frac{1}{2}(l_2 - l_1,)$

$$2R_1 = 2\rho + \frac{l_1 - l_2}{\tan D_1};$$

and by parity of reasoning,

$$2R_2 = 2\rho - \frac{l_1 - l_2}{\tan D_2}.$$

From either of the two sets of relations the curvatures of the two arcs are easily computed.

For problems of railway location the curvatures thus obtained will usually require a slight adjustment to integral values in minutes, which will require a small shifting of A_1, A_2 towards T_1, T_2, and will leave a short, straight piece between the arcs A_1J_1, A_2J_2.

If it be desired to construct the compound curve with the least possible change of radius, we have from the last relations

$$R_1 - R_2 = \tfrac{1}{2}(l_1 - l_2)(\cot D_1 + \cot D_2)$$

$$= \frac{(l_1 - l_2)\sin J}{\cos(D_1 - D_2) - \cos J}.$$

This difference is obviously least when the denominator is greatest; that is, when

$$D_1 = D_2.$$

If the change of curvature, as measured by the defect from unity of the ratio of the radii,

$$\frac{R_1}{R_2} = \frac{\sin(A_2 - D_2)\sin D_2}{\sin(A_1 - D_1)\sin D_1}$$

$$= \frac{\cos A_2 - \cos(A_2 - 2D_2)}{\cos A_1 - \cos(A_1 - 2D_1)}$$

$$= 1 - \frac{\cos A_1 - \cos A_2}{\cos A_1 - \cos(A_1 - 2D_1)},$$

is to be least, we must have

$$2D_1 = A_1; \quad \therefore \quad 2D_2 = A_2;$$

and hence T_1T_2 parallel to A_1A_2.

If T_1A_1, T_2A_2 lie on opposite sides of the chord A_1A_2, the curvature of the second branch must be reversed. We have then

$$I = A_2 - A_1 = 2D_2 - 2D_1,$$

and therefore

$$J = D_2 - D_1 = \tfrac{1}{2}(A_2 - A_1).$$

The locus of J is the circular arc on A_1A_2 with radius

$$r = \frac{c}{2 \sin J}.$$

Since $JT_1 = IA_1 - IT_1$, $\quad JT_2 = - IA_2 + IT_2$, and therefore

$$T_1T_2 = (T_2I - T_1I) + (A_1I - A_2I),$$

the envelope of T_1T_2 is the circular arc which touches IA_1, A_2I produced at M_1, M_2, distant $\frac{1}{2}(l_1 - l_2)$ from I, the radius therefore being

$$\rho = \frac{l_1 - l_2}{2 \tan J},$$

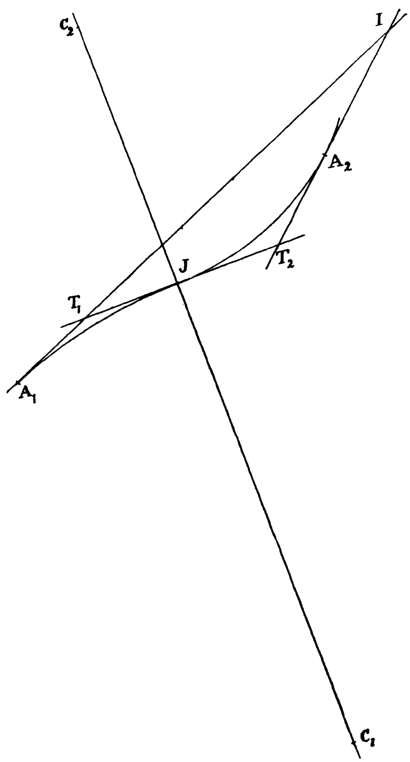

Fig. 3.

As in the previous case, we find

$$\sin \tfrac{1}{2}\Delta_1 = \frac{\sin D_1}{2r \sin (D_2 - A_2)}, \quad \sin \tfrac{1}{2}\Delta_2 = \frac{\sin D_2}{2r \sin (A_1 - D_1)};$$

and

$$2R_1 = \frac{l_1 + l_2}{\tan D_1} - 2\rho, \quad 2R_2 = 2\rho - \frac{l_1 + l_2}{\tan D_2}.$$

The same remarks as to the computation of the curvatures from these formulæ may be repeated here as were made in the former case.

In the special case where the tangents A_1T_1, A_2T_2 are parallel, $A_1 = A_2 = A$, and hence $D_1 = D_2$, $J = 0$, and $r = \infty$. That is, the locus of J is the chord A_1A_2, and hence

$$D_1 = D_2 = A,$$

$$2s_1 + 2s_2 = c.$$

Assuming either s_1 or s_2, the other is known, and we have

$$\sin \tfrac{1}{2}\Delta_1 = \frac{\sin A}{2s_1}, \quad \sin \tfrac{1}{2}\Delta_2 = \frac{\sin A}{2s_2};$$

or

$$R_1 = \frac{s_1}{\sin A}, \qquad R_2 = \frac{s_2}{\sin A}.$$

Conversely

$$R_1 + R_2 = \tfrac{1}{2}c \operatorname{cosec} A,$$

a relation from which either of the radii may be computed when the other is assumed. If it is desired that the curves shall be of equal curvature, we have for the common radius

$$R = \tfrac{1}{4}c \operatorname{cosec} A.$$

To illustrate the foregoing relations, we compute the curvatures of the two branches of a compound curve for which

$$A_1 = 18^\circ\, 42', \quad A_2 = 31^\circ\, 26', \quad c = 11.24.$$

Assuming $D_1 = 10^\circ$, we easily find

$$D_2 = 15^\circ\, 4', \quad A_1 - D_1 = 8^\circ\, 42', \quad A_2 - D_2 = 16^\circ\, 22'.$$

We have then

$\log c$	1.05077
$\log \sin J$	9.62703
$\log 2r$	1.42374

$\log \sin (A_2 - D_2)$	9.44992	$\log \sin (A_1 - D_1)$	9.17973
$\log 2s_1$	0.87366	$\log 2s_2$	0.60347
$\log \sin D_1$	9.23967	$\log \sin D_2$	9.41488
$\log \sin \frac{1}{2}\Delta_1$	8.36601	$\log \sin \frac{1}{2}\Delta_2$	8.81141
$\log \tan D_1$	9.24632	$\log \tan D_2$	9.43007
$\log \sin \frac{1}{2}\Delta_{1c}$	8.36678	$\log \sin \frac{1}{2}\Delta_{2c}$	8.81560
$\log 2t_1$	0.88031	$\log 2t_2$	0.61866
$\log 2t_{1c}$	0.87954	$\log 2t_{2c}$	0.61447
$2t_1$	7.591	$2t_2$	4.156
$2t_{1c}$	7.578	$2t_{2c}$	4.116
$t_1 - t_{1c}$	0.0065	$t_2 - t_{2c}$	0.020
$\frac{1}{2}\Delta_1$	$1^\circ 20' -$	$\frac{1}{2}\Delta_2$	$3^\circ 43' -$
$\frac{1}{2}\Delta_{1c}$	$1^\circ 20'$	$\frac{1}{2}\Delta_{2c}$	$3^\circ 45'$

The notations Δ_{1c}, Δ_{2c}, t_{1c}, t_{2c} are used to signify the corrected values of Δ_1, Δ_2, t_1, t_2 when the curvatures are taken in entire minutes. The gap at J will be

$$0.0065 + 0.02 = 0.0265.$$

If we employed the other set of formulæ, we should find

$$l_1 = 7.637, \quad l_2 = 4.695;$$
$$2\rho = 26.367;$$
$$2R_1 = 43.052, \quad 2R_2 = 15.438;$$
$$\log \sin \tfrac{1}{2}\Delta_1 = 8.36601. \quad \log \sin \tfrac{1}{2}\Delta_2 = 8.81141;$$

as above.

The best guide in selecting the curvature of one branch of the compound curve is obtained by introducing into the foregoing formulæ the lengths n_1, n_2 cut off from the normals A_1C_1, A_2C_2 by the bisectrix of A_1IA_2. It is obvious that

$$n_1 = l_1 \cot J, \quad n_2 = l_2 \cot J;$$

whence we find from the foregoing results

$$n_1 - R_1 = \tfrac{1}{2}(n_2 - n_1)\frac{\cot D_1 - \cot J}{\cot J} = \frac{n_2 - n_1}{2\cos J} \cdot \frac{\sin D_2}{\sin D_1},$$

$$R_2 - n_2 = \tfrac{1}{2}(n_2 - n_1)\frac{\cot D_2 - \cot J}{\cot J} = \frac{n_2 - n_1}{2\cos J} \cdot \frac{\sin D_1}{\sin D_2};$$

and therefore
$$(n_1 - R_1)(R_2 - n_2) = \left(\frac{n_2 - n_1}{2\cos J}\right)^2,$$

which is the desired relation. It serves to determine one radius as soon as the other is given; and shows that if $l_2 > l_1$ (as can always be assumed), then we must have $R_1 < n_1$, $R_2 > n_2$.

Conversely, R_1, R_2 having been determined, we have

$$\cot D_1 = \cot J \frac{n_2 + n_1 - 2R_1}{n_2 - n_1} = \frac{n_2 + n_1 - 2R_1}{l_2 - l_1},$$

$$\cot D_2 = \cot J \frac{2R_2 - n_2 - n_1}{n_2 - n_1} = \frac{2R_1 - n_2 - n_1}{l_2 - l_1};$$

relations that determine the deflections from the radii.

For the reverse curves we obtain the like relations,

$$n_1 + R_1 = \frac{n_2 + n_1}{2\cos J}\frac{\sin D_2}{\sin D_1},$$

$$n_2 + R_2 = \frac{n_2 + n_1}{2\cos J}\frac{\sin D_1}{\sin D_2},$$

$$(n_1 + R_1)(n_2 + R_2) = \left(\frac{n_1 + n_2}{2\cos J}\right)^2,$$

$$\cot D_1 = \frac{n_2 - n_1 - 2R_1}{l_2 + l_1}, \quad \cot D_2 = \frac{n_2 - n_1 + 2R_2}{l_2 + l_1}.$$

It would be easy to deduce these relations from the following theorem analogous to that of page 41 :—

The common normal C_1C_2 envelopes a circle concentric with the circle (ρ) and tangent to the end normals A_1C_1, A_2C_2. The centre of both circles is the intersection point of the bisectrix of I with the bisectrix of the angle N between the normals. The radius of the new circle is $\frac{1}{2}(l_2 - l_2)$.

An interesting problem in railway engineering is furnished by curved turnouts from a curved main line. In this case the inner line of the outer rail of the turnout meets that of the inner rail of the main line at F, the point of the frog. Putting G for the gauge, measured between the inner lines of the rails, and F for the frog angle, we have in the triangle CC_0F

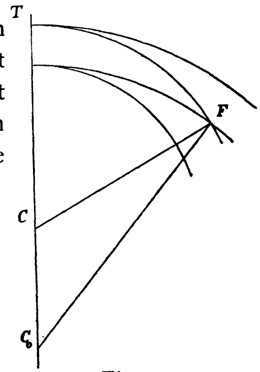

Fig. 4.

$$C = \pi - 2D, \quad C_0 = 2D_0, \quad F = F;$$

$$C_0F = R_0 - \tfrac{1}{2}G, \quad CF = R + \tfrac{1}{2}G, \quad CC_0 = R_0 - R;$$

and by the usual formulæ of trigonometry,

$$\tan D = \frac{G}{2R}\cot\tfrac{1}{2}F,$$

$$\tan D_0 = \frac{G}{2R_0}\cot\tfrac{1}{2}F,$$

$$D = D_0 + \tfrac{1}{2}F.$$

These formulæ enable us to compute all the elements of the turnout when F is given. If R or $\varDelta$ is given, we use the derived formula

$$\tan^2\tfrac{1}{2}F = G(\sin\tfrac{1}{2}\varDelta - \sin\tfrac{1}{2}\varDelta_0 - \sin\tfrac{1}{2}\varDelta\sin\tfrac{1}{2}\varDelta_0)$$

to find F; whence the other elements as before.

If the main track is straight, $\varDelta_0 = 0$, and

$$\tan^2\tfrac{1}{2}F = G\sin\tfrac{1}{2}\varDelta;$$

from which relation either F or $\varDelta$ is known when the other is given.

If the turnout is on the convex side of the main track, it is only necessary to change the sign of R_0 or $\varDelta_0$.